

ENUMERATION OF A DUAL SET OF STIRLING PERMUTATIONS BY THEIR ALTERNATING RUNS

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ABSTRACT. In this paper, we count a dual set of Stirling permutations by the number of alternating runs. Properties of the generating functions, including recurrence relations, grammatical interpretations and convolution formulas are studied.

Keywords: Stirling permutations; Alternating runs; Eulerian polynomials

1. INTRODUCTION

Let $[n] = \{1, 2, \dots, n\}$. The *Stirling number of the second kind* $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ is the number of ways to partition $[n]$ into k blocks. Denote by D the differential operator $\frac{d}{dx}$, and let $\vartheta = xD$. It is well known that

$$\vartheta^n = \sum_{k=1}^n \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} z^k D^k.$$

Let

$$r(x) = \frac{\sqrt{1+x}}{\sqrt{1-x}}.$$

By induction, one can easily verify that there are positive integers $T(n, k)$, $k \in [2n-1]$, such that

$$\vartheta^n(r(x)) = \frac{\sum_{k=1}^{2n-1} T(n, k)x^k}{(1-x)^n(1+x)^{n-1}\sqrt{1-x^2}} \quad \text{for } n \geq 1.$$

It is clear that the numbers $T(n, k)$ satisfy the initial conditions $T(1, 1) = 1$ and $T(1, k) = 0$ for $k \neq 1$. Let $T_n(x) = \sum_{k=1}^{2n-1} T(n, k)x^k$. Using $\vartheta^{n+1}(r(x)) = \vartheta(\vartheta^n(r(x)))$, we get that the polynomials $T_n(x)$ satisfy the recurrence relation

$$T_{n+1}(x) = (2nx + 1)xT_n(x) + x(1 - x^2)T'_n(x) \quad (1)$$

for $n \geq 0$, with the initial values $T_0(x) = 1$ and $T_1(x) = x$. In particular,

$$T_n(1) = -T_{n+1}(-1) = (2n-1)!! \quad \text{for } n \geq 1.$$

Equating the coefficients of x^k on both sides of (1), we get that the numbers $T(n, k)$ satisfy the recurrence relation

$$T(n+1, k) = kT(n, k) + T(n, k-1) + (2n-k+2)T(n, k-2). \quad (2)$$

The motivating goal of this paper is to find a combinatorial interpretation of the numbers $T(n, k)$.

Let \mathfrak{S}_n denote the symmetric group of all permutations of $[n]$, where $[n] = \{1, 2, \dots, n\}$. Let $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$. An *interior peak* in π is an index $i \in \{2, 3, \dots, n-1\}$ such that $\pi(i-1) < \pi(i) > \pi(i+1)$. A *left peak* in π is an index $i \in [n-1]$ such that $\pi(i-1) < \pi(i) > \pi(i+1)$, where we take $\pi(0) = 0$. Let $\text{ipk}(\pi)$ (resp. $\text{lpk}(\pi)$) be the number of interior peaks (resp. left peaks) in π . We say that π changes direction at position i if either $\pi(i-1) < \pi(i) > \pi(i+1)$, or $\pi(i-1) > \pi(i) < \pi(i+1)$, where $i \in \{2, 3, \dots, n-1\}$. We say that π has k *alternating runs* if there are $k-1$ indices i such that π changes direction at these positions. Denote by $\text{altrun}(\pi)$ the number of alternating runs in π .

Define

$$W_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\text{ipk}(\pi)}, \quad \widehat{W}_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\text{lpk}(\pi)}, \quad R_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\text{altrun}(\pi)}.$$

From [14, Corollary 2, Theorem 3], we get

$$\frac{(1+x)^2}{2x} R_n(x) = xW_n(x^2) + \widehat{W}_n(x^2).$$

Let $R_n(x) = \sum_{k=1}^{n-1} R(n, k)x^k$. The study of alternating runs of permutations was initiated by André [2], and he proved that the numbers $R(n, k)$ satisfy the recurrence relation

$$R(n, k) = kR(n-1, k) + 2R(n-1, k-1) + (n-k)R(n-1, k-2) \quad (3)$$

for $n, k \geq 1$, where $R(1, 0) = 1$ and $R(1, k) = 0$ for $k \geq 1$. It follows from (3) that the polynomials $R_n(x)$ satisfy the recurrence relation

$$R_{n+2}(x) = x(nx+2)R_{n+1}(x) + x(1-x^2)R'_{n+1}(x), \quad (4)$$

with the initial value $R_1(x) = 1$. Recall that a *descent* of a permutation $\pi \in \mathfrak{S}_n$ is a position i such that $\pi(i) > \pi(i+1)$. Denote by $\text{des}(\pi)$ the number of descents of π . Then the equations

$$A_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\text{des}(\pi)+1} = \sum_{k=1}^n \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle x^k,$$

define the *Eulerian polynomial* $A_n(x)$ and the *Eulerian number* $\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle$. The polynomial $R_n(x)$ is closely related to $A_n(x)$:

$$R_n(x) = \left(\frac{1+x}{2} \right)^{n-1} (1+w)^{n+1} A_n \left(\frac{1-w}{1+w} \right), \quad w = \sqrt{\frac{1-x}{1+x}}, \quad (5)$$

which was first established by David and Barton [7, 157-162] and then stated more concisely by Knuth [11, p. 605]. There is a large literature devoted to the polynomials $R_n(x)$ (see [22, A059427]). The reader is referred to [4, 14] for recent results on this subject.

In [5], Carlitz introduced $C_n(x)$ defined by

$$\sum_{n=0}^{\infty} \left\{ \begin{matrix} n+k \\ k \end{matrix} \right\} x^n = \frac{C_n(x)}{(1-x)^{2k+1}},$$

and asked for a combinatorial interpretation of $C_n(x)$. Riordan [19] noted that $C_n(x)$ is the enumerator of trapezoidal words with n elements by number of distinct elements, where trapezoidal words are such that the i -th element takes the values $1, 2, \dots, 2i-1$. Gessel and Stanley [8] gave another combinatorial interpretation of $C_n(x)$ in terms of descents of Stirling permutations. A *Stirling permutation* of order n is a permutation $\sigma = \sigma(1)\sigma(2) \cdots \sigma(2n-1)\sigma(2n)$ of the multiset $\{1, 1, 2, 2, \dots, n, n\}$ such that for each i , $1 \leq i \leq n$, all entries between the two occurrences of i are larger than i . Denote by \mathcal{Q}_n the set of Stirling permutation of order n . For $\sigma \in \mathcal{Q}_n$, let $\sigma(0) = \sigma(2n+1) = 0$, and let

$$\begin{aligned} \text{des}(\sigma) &= \#\{i \mid \sigma(i) > \sigma(i+1)\}, \\ \text{asc}(\sigma) &= \#\{i \mid \sigma(i-1) < \sigma(i)\}, \\ \text{plat}(\sigma) &= \#\{i \mid \sigma(i) = \sigma(i+1)\} \end{aligned}$$

denote the number of descents, ascents and plateaux of σ , respectively. Gessel and Stanley [8] proved that

$$C_n(x) = \sum_{\sigma \in \mathcal{Q}_n} x^{\text{des} \sigma}.$$

Bóna [3, Theorem 1] introduced the plateau statistic on \mathcal{Q}_n , and proved that descents, ascents and plateaux are equidistributed over \mathcal{Q}_n . The reader is referred to [9, 10, 12] for recent progress on the study of statistics on Stirling permutations.

In the next section, we show that $T_n(x)$ is the enumerator of a dual set of Stirling permutations of order n by number of alternating runs.

2. COMBINATORIAL INTERPRETATION OF $T(n, k)$

Let $\sigma = \sigma(1)\sigma(2) \cdots \sigma(2n) \in \mathcal{Q}_n$. Let Φ be the injection which maps each first occurrence of entry j in σ to $2j$ and the second j to $2j - 1$, where $j \in [n]$. For example, $\Phi(221331) = 432651$. The *dual set* $\Phi(\mathcal{Q}_n)$ of \mathcal{Q}_n is defined by

$$\Phi(\mathcal{Q}_n) = \{\pi \mid \sigma \in \mathcal{Q}_n, \Phi(\sigma) = \pi\}.$$

Clearly, $\Phi(\mathcal{Q}_n)$ is a subset of \mathfrak{S}_{2n} . For $\pi \in \Phi(\mathcal{Q}_n)$, the entry $2j$ is to the left of $2j - 1$, and all entries in π between $2j$ and $2j - 1$ are larger than $2j$, where $1 \leq j \leq n$. Let ab be an ascent in σ , so $a < b$. Using Φ , we see that ab maps into $(2a - 1)(2b - 1)$, $(2a - 1)(2b)$, $(2a)(2b - 1)$ or $(2a)(2b)$, and vice versa. Note that $\text{asc}(\sigma) = \text{asc}(\Phi(\sigma)) = \text{asc}(\pi)$. Therefore, we have

$$C_n(x) = \sum_{\pi \in \Phi(\mathcal{Q}_n)} x^{\text{asc}(\pi)}.$$

It should be noted that $\pi \in \Phi(\mathcal{Q}_n)$ always ends with a descending run. We now present the following result.

Theorem 1. *We have*

$$T(n, k) = \#\{\pi \in \Phi(\mathcal{Q}_n) \mid \text{altrun}(\pi) = k\}.$$

Proof. There are three ways in which a permutation $\pi \in \Phi(\mathcal{Q}_{n+1})$ with $\text{altrun}(\pi) = k$ can be obtained from a permutation $\sigma \in \Phi(\mathcal{Q}_n)$ by inserting the pair $(2n+2)(2n+1)$ into consecutive positions.

- (a) If $\text{altrun}(\sigma) = k$, then we can insert the pair $(2n+2)(2n+1)$ right before the beginning of each descending run, and right after the end of each ascending run. This accounts for $kT(n, k)$ possibilities.
- (b) If $\text{altrun}(\sigma) = k - 1$, then we distinguish two cases: when σ starts in an ascending run, we insert the pair $(2n+2)(2n+1)$ to the front of σ ; when σ starts in a descending run, we insert the pair $(2n+2)(2n+1)$ right after the first entry of σ . This gives $T(n, k - 1)$ possibilities.
- (c) If $\text{altrun}(\sigma) = k - 2$, then we can insert the pair $(2n+2)(2n+1)$ into the remaining $(2n+1) - (k - 2) - 1 = 2n - k + 2$ positions. This gives $(2n - k + 2)T(n, k - 2)$ possibilities.

Therefore, the numbers $T(n, k)$ satisfy the recurrence relation (2), and this completes the proof. \square

Define

$$M_n(x) = \sum_{\pi \in \Phi(\mathcal{Q}_n)} x^{\text{ipk}(\pi)}, \quad N_n(x) = \sum_{\pi \in \Phi(\mathcal{Q}_n)} x^{\text{lpk}(\pi)}.$$

It follows from [16, Theorem 4] that $M_n(x) = x^n N_n(\frac{1}{x})$. Moreover, from [16, Theorem 5], we have

$$(1 + x)T_n(x) = xM_n(x^2) + N_n(x^2).$$

We now recall some properties of $N_n(x)$. Let $N_n(x) = \sum_{k=1}^n N(n, k)x^k$. Apart from counting permutations in the set $\Phi(\mathcal{Q}_n)$ with k left peaks, the number $N(n, k)$ also has the following combinatorial interpretations:

- (m_1) Let $e = (e_1, e_2, \dots, e_n) \in \mathbb{Z}^n$, and let $I_{n,k} = \{e \in \mathbb{Z}^n \mid 0 \leq e_i \leq (i-1)k\}$, which known as the set of n -dimensional k -inversion sequences (see [20]). The number of *ascents* of e is defined by

$$\text{asc}(e) = \# \left\{ i : 1 \leq i \leq n-1 \mid \frac{e_i}{(i-1)k+1} < \frac{e_{i+1}}{ik+1} \right\}.$$

Savage and Viswanathan [21] discovered that $N(n, k) = \#\{e \in I_{n,2} : \text{asc}(e) = n-k\}$.

- (m_2) We say that an index $i \in [2n-1]$ is an *ascent plateau* of $\pi \in \mathcal{Q}_n$ if $\pi(i-1) < \pi(i) = \pi(i+1)$. The number $N(n, k)$ counts Stirling permutations in \mathcal{Q}_n with k ascent plateaux (see [16, Theorem 3]).
- (m_3) The number $N(n, k)$ counts perfect matching on $[2n]$ with the restriction that only k matching pairs with odd minimal elements (see [18]).

The polynomials $N_n(x)$ satisfy the recurrence relation

$$N_{n+1}(x) = (2n+1)xN_n(x) + 2x(1-x)N'_n(x)$$

with initial value $N_0(x) = 1$. The first few of $N_n(x)$ are

$$N_1(x) = x, N_2(x) = 2x + x^2, N_3(x) = 4x + 10x^2 + x^3, N_4(x) = 8x + 60x^2 + 36x^3 + x^4.$$

The exponential generating function for $N_n(x)$ is given as follows (see [13, Section 5]):

$$N(x, z) = \sum_{n \geq 0} N_n(x) \frac{z^n}{n!} = \sqrt{\frac{1-x}{1-xe^{2z(1-x)}}}. \quad (6)$$

A polynomial $f(x) = \sum_{k=0}^n a_k x^k$ is *symmetric* if $a_k = a_{n-k}$ for all $0 \leq k \leq n$, while it is *unimodal* if there exists an index $0 \leq m \leq n$, such that

$$a_0 \leq a_1 \leq \dots \leq a_{m-1} \leq a_m \geq a_{m+1} \geq \dots \geq a_n.$$

Theorem 2. *The polynomial $T_n(x)$ is symmetric and unimodal.*

Proof. It is immediate from (2) that $T_n(x)$ is a symmetric polynomial. We show the unimodality by induction on n . Note that $T_1(x) = x$, $T_2(x) = x + x^2 + x^3$ and $T_3(x) = x + 3x^2 + 7x^3 + 3x^4 + x^5$ are all unimodal. Thus it suffices to consider the case $n \geq 3$. Assume that $T_n(x)$ is symmetric and unimodal. For $1 \leq k \leq n+1$, it follows from (2) that

$$\begin{aligned} T(n+1, k) - T(n+1, k-1) &= (k-1)(T(n, k) - T(n, k-1)) + (T(n, k-1) - T(n, k-2)) \\ &\quad + (2n-k+2)(T(n, k-2) - T(n, k-3)) + (T(n, k) - T(n, k-3)) \\ &\geq 0, \end{aligned}$$

where the inequalities are follow from the induction hypothesis. This completes the proof. \square

In the next section, we present a grammatical interpretation of $T_n(x)$.

3. GRAMMATICAL INTERPRETATIONS

The grammatical method was introduced by Chen [6] in the study of exponential structures in combinatorics. For an alphabet A , let $\mathbb{Q}[[A]]$ be the rational commutative ring of formal power series in monomials formed from letters in A . A context-free grammar over A is a function $G : A \rightarrow \mathbb{Q}[[A]]$ that replace a letter in A by a formal function over A . The formal derivative D is a linear operator defined with respect to a context-free grammar G . More precisely, the derivative $D = D_G : \mathbb{Q}[[A]] \rightarrow \mathbb{Q}[[A]]$ is defined as follows: for $x \in A$, we have $D(x) = G(x)$; for a monomial u in $\mathbb{Q}[[A]]$, $D(u)$ is defined so that D is a derivation, and for a general element $q \in \mathbb{Q}[[A]]$, $D(q)$ is defined by linearity.

The *hyperoctahedral group* B_n is the group of signed permutations of the set $\pm[n]$ such that $\pi(-i) = -\pi(i)$ for all i , where $\pm[n] = \{\pm 1, \pm 2, \dots, \pm n\}$. For each $\pi \in B_n$, we define

$$\begin{aligned} \text{des}_A(\pi) &:= \#\{i \in \{1, 2, \dots, n-1\} \mid \pi(i) > \pi(i+1)\}, \\ \text{des}_B(\pi) &:= \#\{i \in \{0, 1, 2, \dots, n-1\} \mid \pi(i) > \pi(i+1)\}, \end{aligned}$$

where $\pi(0) = 0$. Following [1], the *flag descent number* of π is defined by

$$\text{fdes}(\pi) := \begin{cases} 2\text{des}_A(\pi) + 1, & \text{if } \pi(1) < 0; \\ 2\text{des}_A(\pi), & \text{otherwise.} \end{cases}$$

Let

$$\begin{aligned} B_n(x) &= \sum_{\pi \in B_n} x^{\text{des}_B(\pi)} = \sum_{k=0}^n B(n, k) x^k, \\ S_n(x) &= \sum_{\pi \in B_n} x^{\text{fdes}(\pi)} = \sum_{k=1}^{2n} S(n, k) x^{k-1}. \end{aligned}$$

The polynomial $B_n(x)$ is called an *Eulerian polynomial of type B*, while $B(n, k)$ is called an *Eulerian number of type B* (see [22, A060187]). It follows from [1, Theorem 4.3] that the numbers $S(n, k)$ satisfy the recurrence relation

$$S(n, k) = kS(n-1, k) + S(n-1, k-1) + (2n-k+1)S(n-1, k-2)$$

for $n, k \geq 1$, where $S(1, 1) = S(1, 2) = 1$ and $S(1, k) = 0$ for $k \geq 3$. The polynomials $S_n(x)$ is closely related to the Eulerian polynomial $A_n(x)$:

$$S_n(x) = \frac{1}{x}(1+x)^n A_n(x) \quad \text{for } n \geq 1,$$

which was established by Adin, Brenti and Roichman [1]. It should be noted that $S_n(x)$ and $A_n(x)$ are both symmetric.

Consider the context-free grammar

$$A = \{x, y, z\}, \quad G = \{x \rightarrow p(x, y, z), y \rightarrow q(x, y, z), z \rightarrow r(x, y, z)\},$$

where $p(x, y, z), q(x, y, z)$ and $r(x, y, z)$ are polynomials in x, y and z . The *diamond product* of z with the grammar G is defined by

$$G \diamond z = \{x \rightarrow p(x, y, z)z, y \rightarrow q(x, y, z)z, z \rightarrow r(x, y, z)z\}.$$

We now recall two results on context-free grammars.

Proposition 3 ([14, Theorem 6]). *If*

$$G = \{x \rightarrow xy, y \rightarrow yz, z \rightarrow y^2\}, \tag{7}$$

then

$$D^n(x^2) = x^2 \sum_{k=0}^n R(n+1, k) y^k z^{n-k}.$$

Setting $x = z = 1$, we have $D^n(x^2)|_{x=z=1} = R_{n+1}(y)$.

Proposition 4 ([15, Theorem 10]). *Consider the context-free grammar*

$$G' = \{x \rightarrow xyz, y \rightarrow yz^2, z \rightarrow y^2z\}, \tag{8}$$

which is the diamond product of z with the grammar G defined by (7). For $n \geq 1$, we have

$$\begin{aligned} D^n(xy) &= x \sum_{k=1}^{2n} S(n, k) y^{2n-k+1} z^k, \\ D^n(yz) &= \sum_{k=0}^n B(n, k) y^{2n-2k+1} z^{2k+1}, \\ D^n(y) &= \sum_{k=1}^n N(n, k) y^{2n-2k+1} z^{2k}, \\ D^n(z) &= \sum_{k=1}^n N(n, n-k+1) y^{2n-2k+2} z^{2k-1}, \\ D^n(y^2) &= 2^n \sum_{k=1}^n \left\langle n \atop k \right\rangle y^{2n-2k+2} z^{2k}. \end{aligned}$$

We can now conclude the following result.

Theorem 5. *Let G' be the context-free grammar given by (8). Then for $n \geq 1$, we have*

$$\begin{aligned} D^n(x) &= x \sum_{k=1}^{2n-1} T(n, k) y^k z^{2n-k}, \\ D^n(x^2) &= 2x^2(y+z)^{n-1} \sum_{k=1}^n \left\langle n \atop k \right\rangle y^k z^{n-k+1}. \end{aligned}$$

Setting $x = z = 1$, we have $D^n(x)|_{x=z=1} = T_n(y)$ and $D^n(x^2)|_{x=z=1} = 2(1+y)^{n-1} A_n(y)$.

Proof. Note that $D(x) = xyz$ and $D^2(x) = xyz^3 + xy^2z^2 + xy^3z$. For $n \geq 1$, we define $t(n, k)$ by

$$D^n(x) = x \sum_{k \geq 1} t(n, k) y^k z^{2n-k}.$$

Then

$$\begin{aligned} D^{n+1}(x) &= D(D^n(x)) \\ &= x \sum_{k \geq 1} t(n, k) y^{k+1} z^{2n-k+1} + x \sum_{k \geq 1} kt(n, k) y^k z^{2n-k+2} + x \sum_{k \geq 1} (2n-k)t(n, k) y^{k+2} z^{2n-k}. \end{aligned}$$

Hence

$$t(n+1, k) = kt(n, k) + t(n, k-1) + (2n-k+2)t(n, k-2). \quad (9)$$

By comparing (9) with (2), we see that the numbers $t(n, k)$ satisfy the same recurrence relation and initial conditions as $T(n, k)$, so they agree. The assertion for $D^n(x^2)$ can be proved in a similar way. \square

It follows from *Leibniz's formula* that

$$D^n(uv) = \sum_{k=0}^n \binom{n}{k} D^k(u) D^{n-k}(v).$$

Hence

$$\begin{aligned} D^n(x^2) &= \sum_{k=0}^n \binom{n}{k} D^k(x) D^{n-k}(x), \\ D^{n+1}(x) &= D^n(xyz) = \sum_{k=0}^n \binom{n}{k} D^k(x) D^{n-k}(yz) = \sum_{k=0}^n \binom{n}{k} D^k(xy) D^{n-k}(z). \end{aligned}$$

Therefore, we can use Proposition 4 and Theorem 5 to get several convolution identities.

Corollary 6. *For $n \geq 1$, we have*

$$2(1+x)^{n-1}A_n(x) = \sum_{k=0}^n \binom{n}{k} T_k(x) T_{n-k}(x), \quad (10)$$

$$T_{n+1}(x) = x \sum_{k=0}^n \binom{n}{k} T_k(x) B_{n-k}(x^2),$$

$$T_{n+1}(x) = x \sum_{k=0}^n \binom{n}{k} S_k(x) N_{n-k}(x^2).$$

Let $T(x, z) = \sum_{n=0}^{\infty} T_n(x) \frac{z^n}{n!}$. Recall that the exponential generating function for $A_n(x)$ is given as follows (see [22, A008292]):

$$A(x, t) = \sum_{n \geq 0} A_n(x) \frac{t^n}{n!} = \frac{1-x}{1-xe^{t(1-x)}}. \quad (11)$$

Combining (10) and (11), we get

$$T(x, z) = \frac{e^{z(x-1)(x+1)} + x}{1+x} \sqrt{\frac{1-x^2}{e^{2z(x-1)(x+1)} - x^2}}. \quad (12)$$

From (6), we have

$$\sum_{n \geq 0} M_n(x^2) \frac{z^n}{n!} = \sum_{n \geq 0} x^{2n} N_n \left(\frac{1}{x^2} \right) \frac{z^n}{n!} = \sqrt{\frac{1-x^2}{e^{2z(x-1)(x+1)} - x^2}}.$$

Note that

$$\frac{e^{z(x-1)(x+1)} + x}{1+x} = 1 + \sum_{n \geq 1} (x-1)^n (x+1)^{n-1} \frac{z^n}{n!}.$$

Therefore, from (12), we obtain

$$T_n(x) = M_n(x^2) + \sum_{k=0}^{n-1} \binom{n}{k} M_k(x^2) (x-1)^{n-k} (x+1)^{n-k-1} \quad \text{for } n \geq 1.$$

4. CONCLUDING REMARKS

In this paper, we show that the polynomial $T_n(x)$ has many similar properties to $R_n(x)$. In fact, there are more similar properties deserve to be studied. From the relation (5) and the fact that $A_n(x)$ have only real zeros, Wilf [23] proved that $R_n(x)$ have only real zeros for $n \geq 2$. Moreover, it follows from (4) that all zeros of $R_n(x)$ belong to $[-1, 0]$, and the zeros of $R_n(x)$ separate that of $R_{n+1}(x)$ (see [17, Corollary 8]).

Let $f(x)$ and $F(x)$ be two polynomials with only real coefficients. Suppose that $f(x)$ and $F(x)$ both have only imaginary zeros. We say that $f(x)$ *separates* $F(x)$ if $\deg F = \deg f + 2$ and the sequences of real and imaginary parts of the zeros of $f(x)$ respectively separate that of $F(x)$. In other words, let $f(x) = a \prod_{j=1}^{n-1} (x + p_j + q_j i)(x + p_j - q_j i)$, and let $F(x) = b \prod_{j=1}^n (x + s_j + t_j i)(x + s_j - t_j i)$, where a, b are respectively leading coefficients of $f(x)$ and $F(x)$, $p_1 \geq p_2 \geq \dots \geq p_{n-1}$, $q_1 \geq q_2 \geq \dots \geq q_{n-1}$, $s_1 \geq s_2 \geq \dots \geq s_n$ and $t_1 \geq t_2 \geq \dots \geq t_n$. Then we have

$$s_1 \geq p_1 \geq s_2 \geq p_2 \geq \dots \geq s_{n-1} \geq p_{n-1} \geq s_n,$$

$$t_1 \geq q_1 \geq t_2 \geq q_2 \geq \dots \geq t_{n-1} \geq q_{n-1} \geq t_n.$$

Based on empirical evidence, we propose the following conjecture.

Conjecture 7. *For $n \geq 2$, all zeros of $T_n(x)/x$ are imaginary and $T_n(x)/x$ separates $T_{n+1}(x)/x$.*

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